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Universal scaling of Lyapunov exponents

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Abstract. We prove numerically for the first time that the lower part of the spectrum of the transfer matrix of the quasi-one-dimensional disordered system is strongly correlated in the neighbourhood of the critical point of the metal–insulator transition. In particular, the disorder and the system size dependence of the spectrum is governed only by one parameter which we identify with the scaling parameter of MacKinnon and Kramer. The strong correlation of the spectrum supports the idea of the universality of the random-matrix-like theory of the metal–insulator transition and provides us with a new way of calculation of the critical parameters of this transition.

In spite of the great success in the theoretical and numerical studies of the disorder-induced metal–insulator transition (MIT), no general theory of this phenomena has yet been formulated. The most objective attempt to describe MIT in a unified manner is based on the analysis of the spectrum of the transfer matrix. This treatment is motivated by the successful application of the random matrix theory (RMT) [1] to the description of the transport properties of the disordered *metallic* samples [2–5]. For the transfer matrix, T, we introduce 'extended' Lyapunov exponents, ν_i, through eigenvalues, Λ_i, of the matrix $T^\dagger T$ as $\nu_i = \log \Lambda_i$ [8]. Then, observed linearity of the spectrum of $\nu_i \propto i$ in the metallic regime, inspired a search of their distribution in a form

$$P(\nu_1, \nu_2, \ldots, \nu_N) = \exp -\beta \mathcal{H}(\nu_1, \nu_2, \ldots, \nu_N) \tag{1}$$

with the Hamiltonian

$$\mathcal{H} = \sum_i V(\nu_i) + \sum_{i<j} u(\nu_i, \nu_j). \tag{2}$$

Here, $N = L^{d-1}$ is a number of channels in the quasi-one-dimensional (Q1D) system $L^{d-1} \times L_z$ and $\beta = 1$, 2 or 4 depending on the symmetry of the problem [1, 3].

Distribution (1) has been approved by applying the local and global maximum entropy ansatz [6]. Although this analysis was addressed only to systems with very small disorder (metallic limit), there is evidence [7–9] that the RMT-like distribution, $P(z)$, could be relevant also at higher values of disorder, including the critical point of MIT and even the localized regime. If this hypothesis is true, then the spectrum of LE should be strongly correlated in the sense that the i-dependence of the ith LE ν_i should follow a simple formula similar to the linear relation $\nu_i \propto i$ in the metallic limit. The transition from the metallic into the localized state would then be caused by the appropriate change of the one-particle potential, $V(\nu)$, and/or of the interacting term $u(\nu_i, \nu_j)$. The form of these potentials has

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to guarantee the most important features of the spectrum of LE, namely (i) the linearity of the spectrum in the metallic regime and (ii) the opening of the gap at the critical point.

In the metallic regime, the one-particle potential $V(\nu)$ is always quadratic. Recently another potential $V(\nu)$ has been proposed [10] for which the model (1) is *exactly solvable* and for which the spectrum of LE has the above-mentioned desired properties. In a previous work [9] we have taken another way. We have analysed the numerical data for the Q1D Anderson model and proposed the form of potential $V(\nu)$ which reproduces the spectrum of LE. In the Q1D systems the interacting potential always has a simple universal form

$$u(\nu_i, \nu_j) = \tfrac{1}{2} \log |\cosh \nu_i - \cosh \nu_j|$$

independent from the regime system in [9, 11]. Consequently, it is the one-particle potential which is responsible for the transition from metallic into the localized regime. Although the model we have obtained is neither explicitly derived nor exactly solvable, it reproduces correctly the disorder and dimension dependence of the spectra in all three regimes (metallic, critical, localized) and was even able to explain the quantitative differences between spectra of Q1D and cubic systems reported in [8].

The important consequence of the universal RMT-like theory of MIT is the strong correlation of the spectrum of LE. Up to now, there is no direct evidence that the spectrum of LE close to the critical point is really strongly correlated. The most successful numerical method of analysis of MIT, the finite-size scaling theory of localization [12], deals only with the smallest positive LE ν_1. The first attempt to analyse the scaling behaviour of higher LE was made in [13]. Evidence that at least the lower part of the spectrum of LE at the critical point could be described by simple formula which contains no more than one parameter (the value of the first LE z_1) were presented in [8] for three-dimensional samples and in [14, 15, 9] for four-dimensional systems. The aim of this paper is, therefore, the detailed study of the spectrum of the Q1D disordered system. As a model, we use the Anderson model, which undergoes transition from the metallic state into the localized one when the strength of the disorder, W, represented by random energies on the sites of the lattice, exceeds critical value $W_c \approx 16.5$.

In studies of the spectrum of LE, we use numerical data for the Q1D Anderson model $L^2 \times L_z$ ($L_z \gg L$), collected by Henneke [13] for $8 \leqslant L \leqslant 14$ and completed by ourselves for $L = 16$. In the limit $L_z \gg L$, all ν_i are proportional to the system length L_z and so $\nu_i \gg 1$. This simplifies the analysis of the spectrum of ν's [9]. In what follows we consider LE z_i, defined as as $z_i = \frac{L}{L_z}\nu_i$. They depend only on L and on the disorder W.

We concentrate on the lower part of the spectrum, in which the changes of the spectrum are more visible and consider only z_i with $i \leqslant N_{\mathrm{LE}}$. The choice of the number of LE and its L-dependence of N_{LE} ($N_{\mathrm{LE}} \propto L$) will be discussed below.

For any system size L we find that the N_{LE} smallest Lyapunov exponents in the neighbourhood of the critical point fulfil the simple formula (5). Moreover, this relation is determined by only one parameter, which we identify with the scaling parameter, $\xi(W)$. The latter was introduced by MacKinnon and Kramer [12] in the scaling theory of localization, where it defines the disorder and the system size dependence of the first LE:

$$z_1 = f(\xi(W)/L).$$

The fact that the same scaling parameter determines the behaviour of not only the first LE, but also of the higher ones, is of extreme importance for the formulation of the general theory of MIT.

Figure 1 shows the typical spectra of LE at the critical point $W_c = 16.5$ and in the

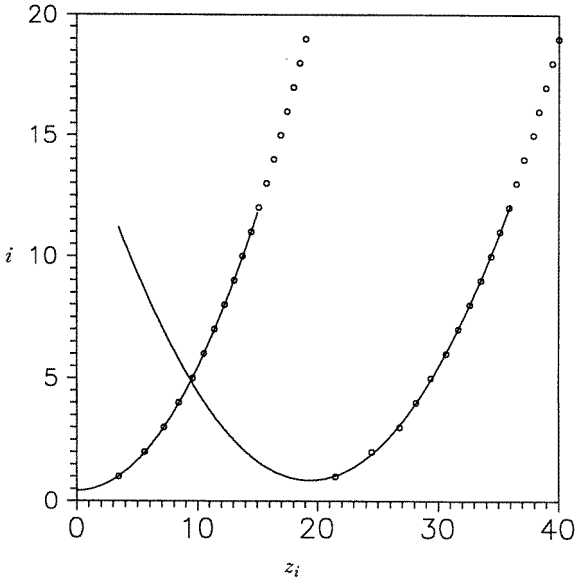

Figure 1. Spectrum of LE i versus z_i for $L = 16$ and $W = 16.5$ (critical point) and $W = 32$ (insulator) together with corresponding fits (3).

localized regime $W = 32$. The lower part of the spectrum can be fitted with quadratic fit

$$i - 1 = a_0 + a_1 z_i + a_2 z_i^2 \tag{3}$$

with $a_1 = 0$ at the critical point [8].

The choice of fit (3) has been inspired by the phenomenological description of MIT, proposed recently in [9]: we conjectured that MIT is accompanied by the change of the density $\rho(z) = \langle \sum_i \delta(z - z_i) \rangle$ of LE. In difference to the previous conjecture [9], we consider here the more general z-dependence of $\rho(z)$:

$$\rho(z) = \frac{2}{z_1^3 c(W, L)} [z + a(W, L)]. \tag{4}$$

From (4) we derive, by the method explained in [9], that the LEs of Q1D system fulfil the relation

$$[z_i + a(W, L)]^2 - [z_1 + a(W, L)]^2 = \frac{z_1^3}{2} c(W, L)(i - 1). \tag{5}$$

Relation (5) is equivalent to (3) when $a(W, L) = a_1/(2a)$ and $c(W, L) = 2/z_1^3 a_2$. Comparison with (5) defines the mutual correlation of the parameter of the fit, namely

$$a_0 = -(z_1^2 a_2 + 2 z_1 a_1). \tag{6}$$

Relation (5) assures the strong correlation of the spectrum of LE mention above: the data for LE z_i, completed for different system widths, L, and different disorders, W, in all three regimes are determined by the value of the first LE z_1 and the functions $a(W, L)$ and $c(W, L)$. In fact we will show that the correlation is even stronger: the spectrum is completely determined only by one parameter—the ratio of the scaling parameter $\xi(W)$ [12] to the system width L.

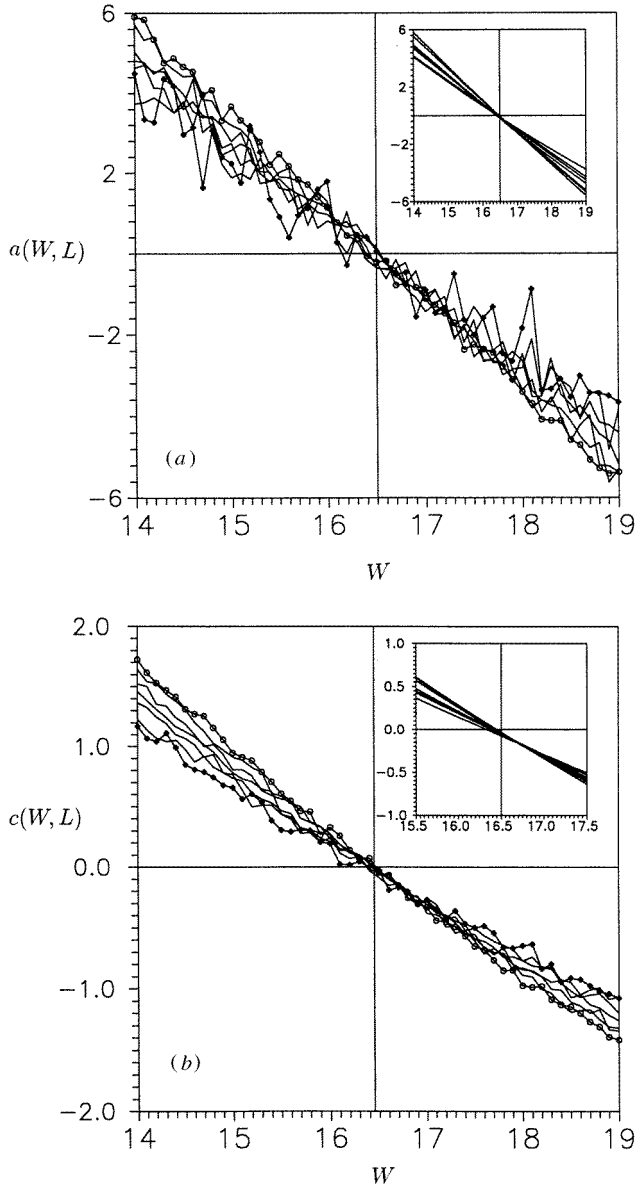

Figure 2. The disorder dependence of (*a*) $a(W, L)$ (*b*) $\tilde{c}(W, L)$ for $L = 8$–14. Inset: quadratic fits throughout the data.

According to formula (5), the most important change of the spectrum of LE is caused by change of the sign of $a(W, L)$ at the critical point. As will be seen below (see figure 2) $a(W, L)$ is linear at the critical point:

$$a(W, L) \propto W - W_c. \tag{7}$$

Far from the critical point $a(W, L)$ behaves as $\propto \pm 2L/\xi(W)$ where the sign $+$ $(-)$ correspond to the metallic (localized) regime, respectively [9]. Then the formula (5) reproduces correctly the linearity of the spectrum in the metallic regime ($a \gg 1$) and

explains the peculiarities of the spectrum in the localized regime.

Relation (7) is also supported by numerical evidence that the spectrum of LE at the critical point fulfils the simple relation [8, 14, 9]

$$z_i = z_1\sqrt{1 + z_1/2(i-1)}$$

which coincides with (5) iff

$$a(W_c, L) = 0 \tag{8}$$

and

$$c(W_c, L) = 1 \tag{9}$$

for each system size L.

First, we have calculated the quadratic fits (3) of the spectrum of LE for different values of disorder $14 \leqslant W \leqslant 19$ and found *for each system size* $8 \leqslant L \leqslant 16$ *separately* the critical disorder, W_c, and the optimal number of LE, N_{LE}, for which the relations (8) and (9) are fulfilled with the best accuracy. Results are collected in table 1 and confirm the linear growth of the optimal number of LE, N_{LE}, with the system size. The estimated value of critical disorder, $16.4 \leqslant W_c \leqslant 16.5$, is in very good agreement with previously obtained data [12, 16, 17, 19] The data in the fifth column of table 1 seem to support the relation $z_1(W_c) = (2\pi)^{2/3} = 3.405\ldots$, which was predicted previously [9] from the comparison of the spectra of LE for cubic and Q1D systems in the *metallic* regime.

Figure 2 presents the disorder dependence of $a(W, L)$ and $\tilde{c}(W, L) = \log c(W, L)$ for different system widths, L, as were calculated from the quadratic fit (3). The uncertainty of the data is rather large and is caused by the amplification of the numerical uncertainty of the raw data for LE. To test the quality of the fit (3) we compare coefficient a_0 obtained from the fit procedure with that predicted by relation (6) we found that the relative deviation of a_0 from its 'theoretical' value (6) does not exceed $1 - 2\%$ (table 1). Another test consists of the study of the difference $\Delta z = z_i - z_i^{\mathrm{fit}}$ between the numerical data z_i and values z_i^{fit} obtained from (3) and its comparison with the inacuracy δz of LEs as was obtained from numerical simulations [13]. In figure 3 we present the ratio $\Delta z/\delta z$ for three different values of disorder. Results confirm that the difference between numerical data and data obtained from the fit are smaller or of the order of the accuracy of the data itself.

Our results confirm an assumption that both $a(W, L)$ and $\tilde{c}(W, L)$ change the sign at the critical point (see equations (8) and (9)). Moreover, all curves seem to have a tendency to cross at the same point very close to the critical point W_c. This tendency is more visible, when we use the quadratic fit through the data for a, c instead of data themselves. It indicates

Table 1. The values of N_{LE}, W_c and z_1, found from the quadratic fit of (3). Data in the last column shows how relation (6) is fulfilled at the critical point ($a_1 = 0$).

L	N_{LE}	W_c	z_1	$z_1/(2\pi)^{2/3}$	$-z_1\sqrt{a_2/a_0}$
8	5	16.419	3.393	0.997	1.004
9	6	16.461	3.416	1.003	1.011
10	7	16.468	3.426	1.006	1.014
11	8	16.421	3.400	0.999	1.016
12	9	16.418	3.379	0.992	1.015
13	9	16.498	3.442	1.011	1.015
14	10	16.485	3.431	1.008	1.014
16	12	16.462	3.412	1.002	1.019

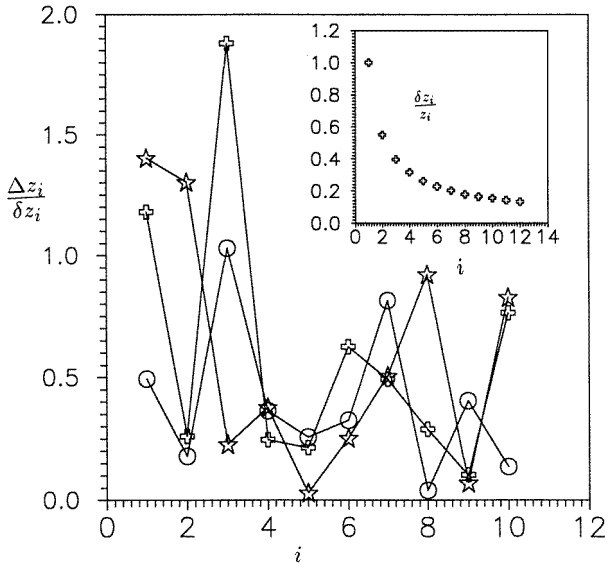

Figure 3. The ratio $\Delta z/\delta z$ for $W = 16$ (circles), $W = 16.5$ (stars) and $W = 17$ (crosses) shows that the difference Δz between numerical and fitted values of LE is smaller or of the order of the numerical accuracy δz of z's. Inset: relative accuracy $\delta z/z$ (in %) of the numerical data for the first 10 LE ($L = 14$, $W = 16.5$) [13].

that functions $a(W, L)$, $\tilde{c}(W, L)$ depend only on one parameter. It is then reasonable to assume that

$$a(W, L) = a(\xi(W)/L) \qquad \tilde{c}(W, L) = \tilde{c}(\xi(W)/L). \tag{10}$$

To prove this relation we plot (10) in figure 4 using the scaling parameter $\xi(W)$ calculated for z_1 in [13] to prove relations (10).

It would be, in principle, possible to use our data for $a(W, L)$ and $\tilde{c}(W, L)$ also for calculation of the scaling parameter $\xi(W)$ and the critical exponents s and ν for the conductance and for the localization length, respectively, in the same way as it was made in the finite-size-scaling analysis on the basis of the data for the first LE. However, results of such calculations are very sensitive to the accuracy of the input data. (The accuracy of the LE used in our analysis, is given in the inset of figure 3 [13].) It is why, in figure 4, we used the scaling parameter found from the studies of the first LE z_1. We present here only the estimation of the critical exponents based on the study of the scaling properties of $a(W, L)$. Following scaling arguments [18] we find that $a(W, L)$ behaves close to the critical point as

$$a(W, L) = A \times (W - W_c)L^{\alpha} \tag{11}$$

with the exponent $\alpha^{-1} = s = \nu$. We fit our data for $a(W, L)$ to formula (11) by the method described in [17] and found

$$1.3 \leqslant s, \nu \leqslant 1.9. \tag{12}$$

It is probably impossible to improve this estimation with the presently available data for LEs. Nevertheless, this result is consistent with the previous estimation $s = \nu \approx 1.5$ [17, 13, 18, 19].

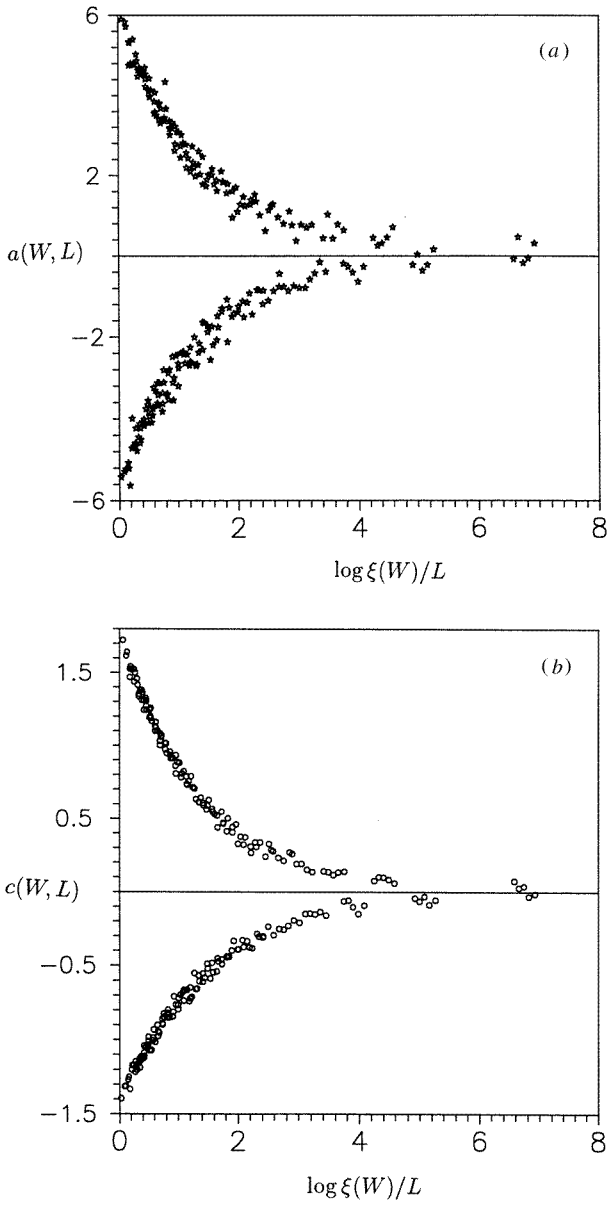

Figure 4. Functions (*a*) $a(W, L)$ and (*b*) $\tilde{c}(W, L)$ as a function of $\xi(W)/L$ with the scaling parameter $\xi(W)$ found from the scaling analysis of the first Lyapunov exponent [13].

In conclusion, we have shown that the system size and disorder dependence of the most relevant part of the spectrum of LE is controlled only by one parameter: the ratio $\xi(W)/L$. This indicates the possibility of generalizing the finite-size scaling theory of localization to the higher LE. Although the present study is probably of little use in practical calculation of the parameters of the transition, the proof of the simultaneous scaling of LE leads to a deeper understanding of the scaling of localization: in the metallic limit, the identification of ξ^{-1} with the conductance of the system requires implicitly that higher LE z_i scale as the

functions $\xi(W)/L$ as it follows explicitly from (5). In the localized regime, the correlation of LE given by relation (5) explains statistical properties of the LE and of the conductance [9, 20].

Our results support the possibility of describing the distribution of LE in the forms (1) and (2) in all three regimes. Their generalization to the d-dimensional system is straightforward [9]. The most important question which remains to be solved is that of the number of LE which has to be considered in the description of MIT. In difference to the treatment of Muttalib *et al* [10], we considered here only a small part of the spectra (the number of considered LE is $N_{\mathrm{LE}} \sim L$). Although it would be alluring to generalize our formulae for the whole spectrum ($N_{\mathrm{LE}} = L^{d-1}$), we did not succeed in doing so. On the other hand, such generalization is probably of little practical importance since larger LE surely play no role in transport and in processes connected with MIT. We believe that present results will inspire the search of the general form of the distribution $P(z)$ of LE.

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